University of North Georgia
Department of Mathematics

Instructor: Berhanu Kidane

Course: Precalculus Math 1113

Text Books: For this course we use free online resources:

See the folder Educational Resources in Shared class files

1) http://www.stitz-zeager.com/szca07042013.pdf (**Book1**)
2) Trigonometry by Michael Corral (**Book 2**)

Other online resources:

Tutorials:

- http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm
- http://archives.math.utk.edu/visual.calculus /
- http://www.ltcconline.net/greenl/java/index.html
- http://en.wikibooks.org/wiki/Trigonometry
- Animation Lessons: http://flashytrig.com/intro/teacherintro.htm
- http://www.sosmath.com/trig/trig.html

Test worksheet generator for Mathematics Teachers

- https://www.kutasoftware.com/

For more free supportive educational resources consult the syllabus

Trigonometric functions (Page 693)

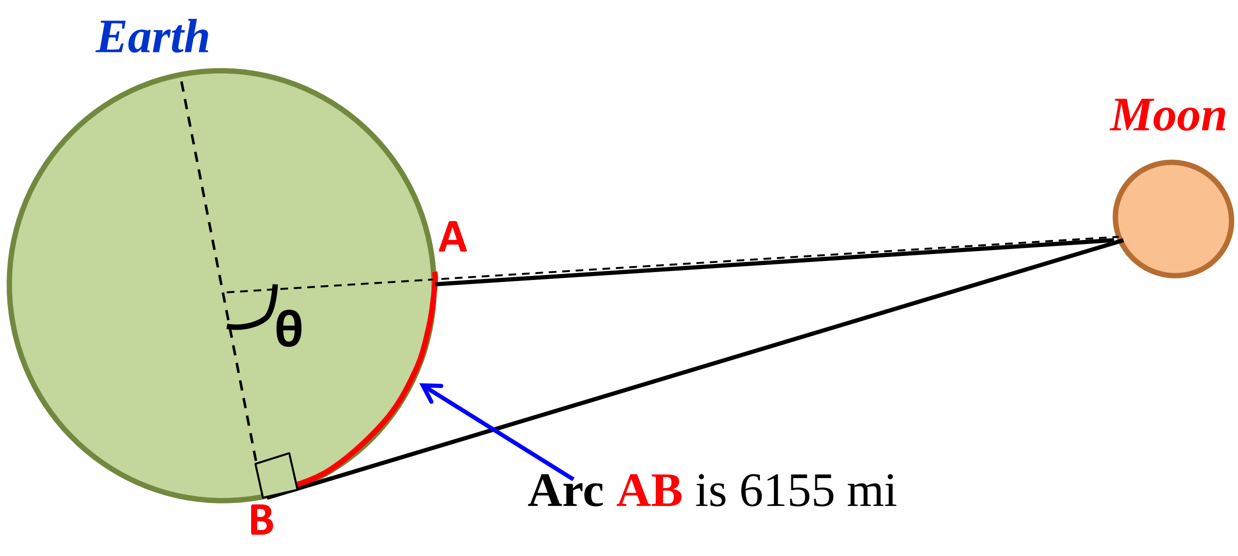
(Book 1)Chapter 10

Objectives: By the end of these chapters Students should be able to:

- Identify degree measures and radian measures
- Convert degree measures in to radian measures and conversely
- Identify co-terminal angles, quadrantal angles, and special angles
- Find trigonometric ratios of angles
- Solve triangles
- Solve application problems
- Work with trigonometric identities

Motivation:

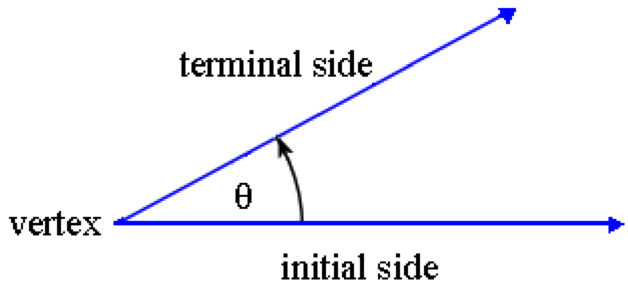
1) **Distance to the Moon:** When the moon is seen at the **zenith** at a point **A** on Earth, it is observed to at the **horizon** from point **B**. Point **A** and **B** are **6155** mi apart, and the radius of the earth is **3960** mi.
   a) Find the angle θ in degree
   b) Estimate the distance from point **A** to the moon i.e. distance from the earth to the moon



2) **Angle of Elevation:** The angle of elevation of an airplane is 23°. If the airplane's altitude is 2500 m, how far away is it?

10.1 Angles and Their Measures

Definition: An **angle** is a measure of the amount of rotation between **two rays** (or line segments). The **two rays** (or line segments) are called the **initial side** and **terminal side. See** the diagram below.

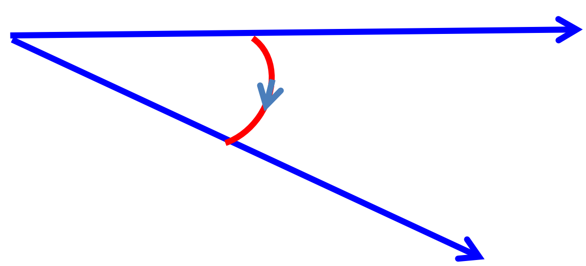


Sign Conventions on Measures of Angles

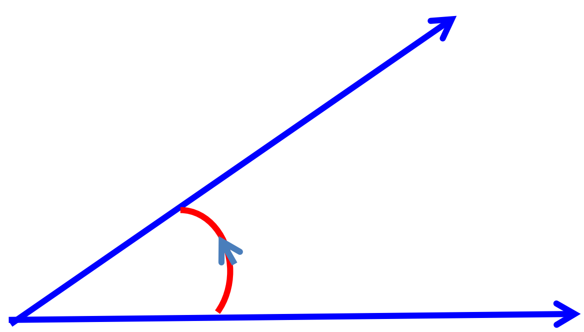
- **Anti-clockwise rotations** have **positive measures**
- **Clockwise rotations** give **negative measures**

Example 1:

A. **Clockwise**, **negative angle**



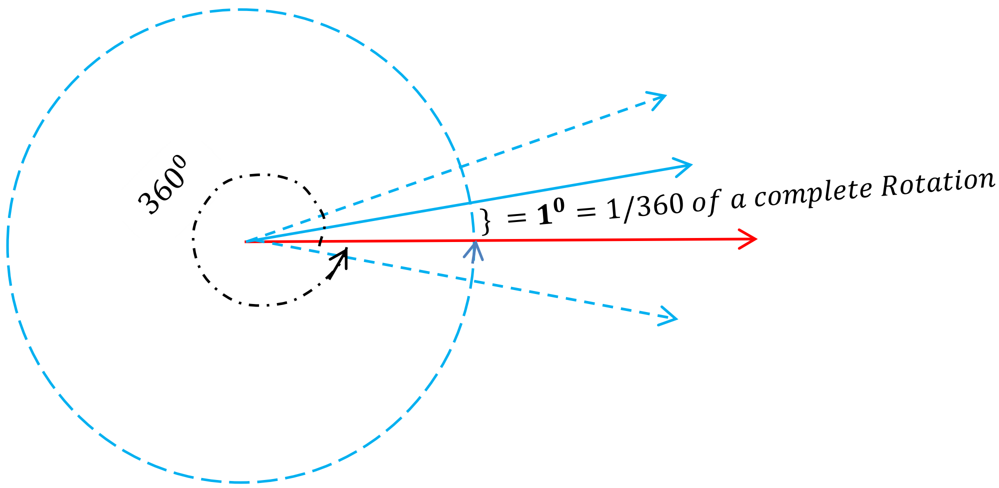
B. Anti-clockwise, **positive angle**



Angles are commonly measured in **degrees** or **radians.**

Degree Measures

One complete rotation has assigned a measure of $\mathbf{360}\ \boldsymbol{degree}, denoted, = \mathbf{360^0;}$ so, $\mathbf{1^0}$ is formed by rotating the initial side $1/360$ *of a complete rotation*. See figure below



Degree, Minutes and Seconds (DMS)

$$1^0 = 60'\ (60\ minutes)$$

$$1' = 60"\ (60\ seconds)$$

Example 1: a) $42.125^0 = 42^0 \&\ 0.125 \times 60' = 42^0 \& 7{,}5' = 42^0 7' \& 0.5 \times 60" = 42^0 7' 30"$

b) $117^0 15' 45" = 117^0 \& 15/60 \& 45/3600 = 117 + 0.25 + 0.0125^0 = 117.2625^0$

Example 2: Worked out Example (Reading HW): Page 696 **Example 10.1.1**

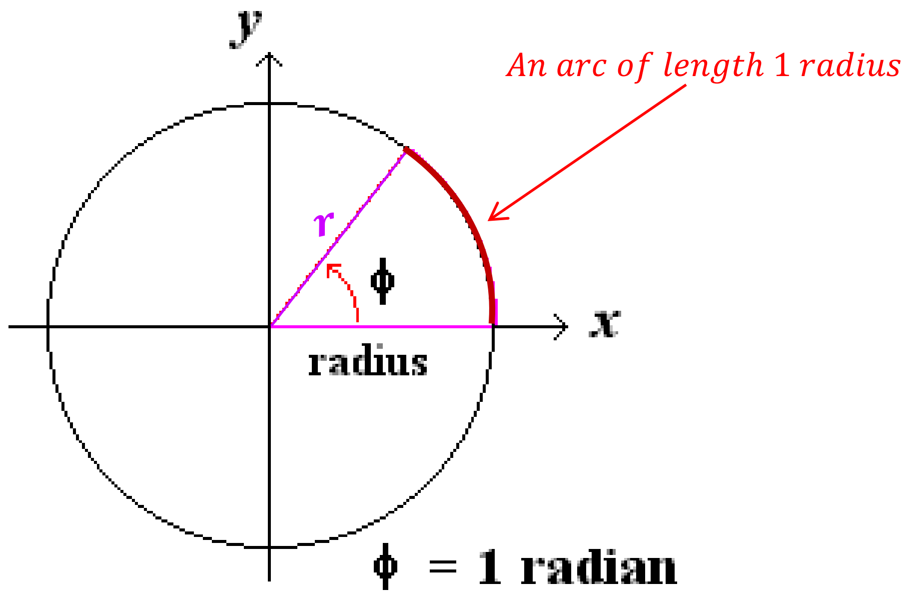
Acute, Right, Obtuse and Straight Angles

- An angle is acute if its measure is between 0^0 and 90^0.
- An angle is a right angle if its measure equals 90^0.
- An angle is obtuse if its measure is between 90^0 and 180^0
- An angle is a straight angle if its measure equals 180^0.

Radian Measures

One radian is defined as the **angle between 2 radii** of a **circle** where the **arc** between them has **length** of **one radius**. In other words: "a **radian** is the **angle subtended** by an **arc** of **length = r** (**the radius**)".

One radian is about **57.3°.**



Given a circle centered at the origin in the Cartesian plane, imagine taking a **radius** and laying it along the **outside circle**, beginning at the ***x axis*** and going counterclockwise, see fig. above (red). This marks out an angle of **one radian**. Because the **circumference** of a circle is **twice** the **radius times pi** (that is, $C = 2r\pi$), a **full circle** corresponds to an angle of 2π **radians**. Thus we get the following **correspondences** between **degree** measure and **radian** measure:

Based on the above definition we see that:

- A central angle of 90^0 cuts off an arc of length $\frac{1}{2}\pi r$; so $90^0 = \frac{1}{2}\pi\ radiuses$ or $90^0 = \frac{1}{2}\pi$ radians
- A central angle of 180^0 cuts off an arc of length πr; so, $180^0 = \pi$ radians
- A central angle of 360^0 cuts off an arc of length 2πr, which is the same as the circumference of the circle. $360^0 = 2\pi$ radians

Relationship between Radian and Degree Measure

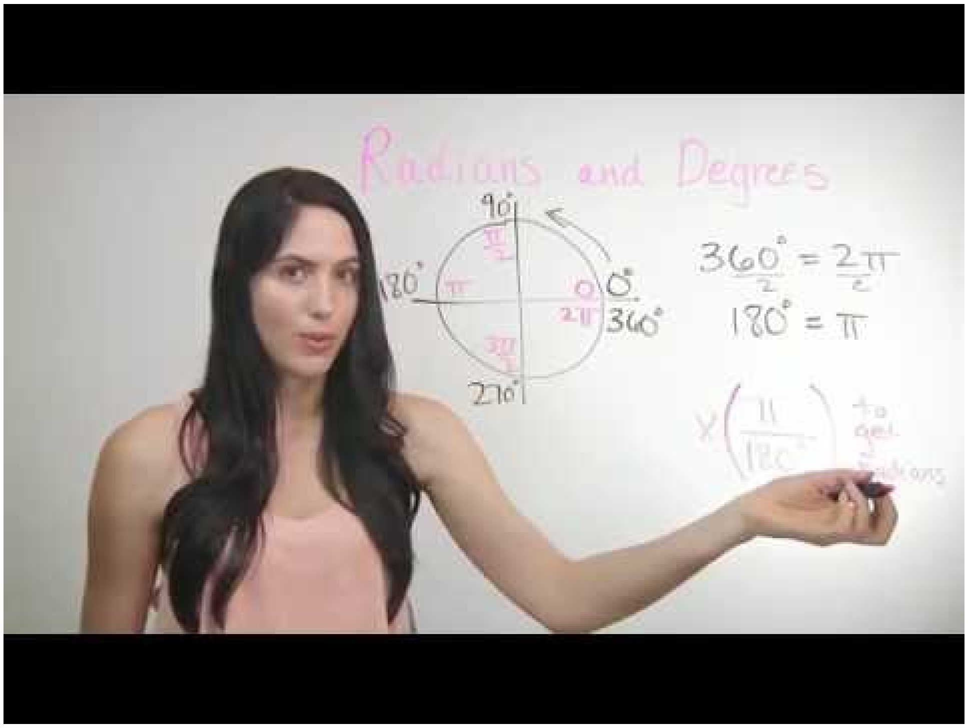
$\mathbf{180^0 = \pi\, rad}$ implies $\left(\frac{180}{\pi}\right)^0 = \mathbf{1\, rad}$ and $\mathbf{1^0 = \frac{\pi}{180}\, rad}$ **,** thus**:**

- To convert degrees to radians, multiply by $\frac{\pi}{180}$
- To convert radians to degrees, multiply by $\frac{180}{\pi}$

Or we have the following conversion formula

$\boxed{S = \frac{\pi\,\theta}{180}}$ **,** where S is **radian measure** and θ is **degree measure**

Example YouTube Vedio.



Example 3:

1. Convert the following to degrees:

a) 1 radian b) 2 radians c) $\frac{3}{2}\pi\, rad$

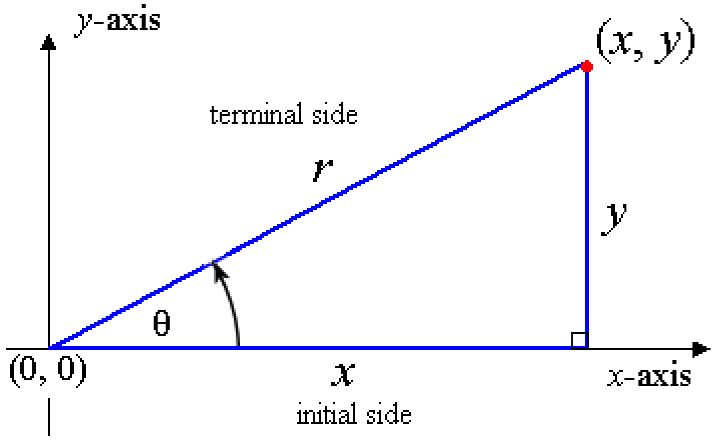
d) $\frac{7}{3}\pi\, rad$ e) $-\frac{1}{3}\pi\, rad$ f) 3.1rad

2. Convert the following to radians:

a) 50° b) 357° c) 60°

d) 156.34° e) $-600°$ f) 1000^0

Standard Position of an Angle

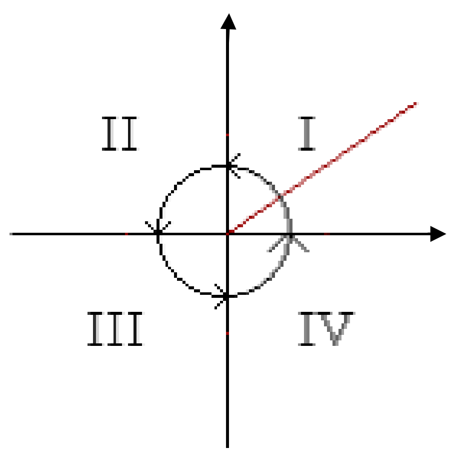


Definition: An angle is in **standard position** if the **initial side** is the **positive x-axis** and the **vertex** is at the **origin.** Figure above angle **θ** is in standard position.

Note: We will use ***r***, the **length** of the **hypotenuse**, and the lengths ***x*** and ***y*** when defining the **trigonometric ratios** or **trigonometric functions** later.

The Four Quadrants

The x and y axes divide the coordinate plane in to four quadrants, denoted, in the counter clockwise direction, by I, II, II, and IV, see figure.



Example: Worked Example (HW Reading) Page 699 Example 10.1.2

Problem 1: In which quadrant does each angle terminate?

a) $15°$ **I** b) $-15°$ c) $135°$

d) $390°$ e) $-100°$ f) $-460°$ g) $710°$

Problem 2: In which quadrant does each angle terminate? (The angles are given in radian measures)

a) $\frac{2}{3}\pi$ b) $\frac{7}{5}\pi$ c) -3.5 d) 11

Co –Terminal Angles

Definition: **Coterminal angles** are **angles** in standard **position** that have a **common terminal side**. For example; 30°, –330° and 390° are all coterminal angles.

Example 1: Find a positive and a negative angle coterminal with a 55° angle also indicate the angles by graphs.

Ans. –305° (= 55 - 360) and 415° (= 55 + 360)

Example 2: Find a positive and a negative angle coterminal with $-\frac{\pi}{3}$ and sketch graphs.

Example 3: Name a non-negative angle that is **coterminal** with each of these, and is less than 360°.

a) 2π b) $450°$ Ans. **90°**

c) $-20°$ d) $-\pi$

e) $-270°$ f) $\frac{7}{3}\pi$

Example 4: Let θ be an angle of the 1st Quad. Give a formula in degrees as well as in radians for the **coterminal** angles of θ**.**

Example: Worked Example (HW Reading) Page 702 **Example 10.1.3**

Quadrantal Angles

Definition: A **Quadrantal angle** is an angle that **terminates** *on* the ***x*- or *y*-**axis.

Example 5: a) What are the quadrantal angles in degrees?

0°, 90°, 180°, 270°, and angles coterminal with them.

b) What are the quadrantal angles in radians?

0, π/2, π, 3π/2, and angles co-terminal with them

Special Angles

Definition: **Special angles** are angles in the **standard position** that are either **Quadrantal angles** or angles that have the **same terminal side** as that of:

30°, 45°, 60°, 120°, 135°, 150°, 210°, 225°, 240°, 300°, 315°, and 330°.

In terms of radian measures

Special angles are angles in the standard position that are either **Quadrantal angles** or angles that have the **same terminal side** as that of:

$$0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}$$

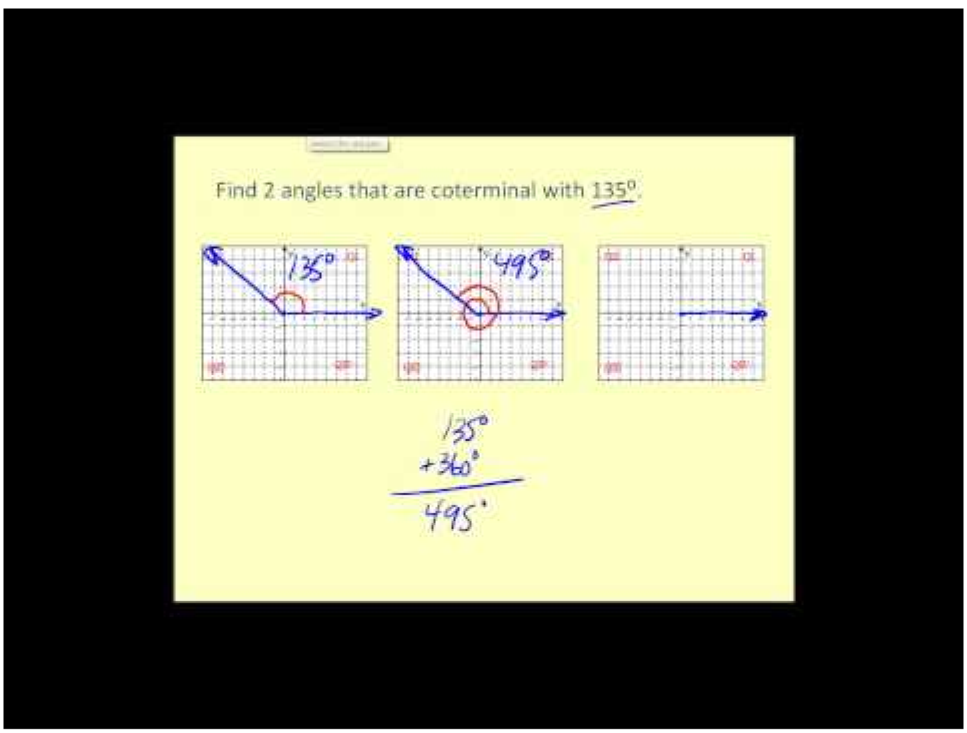
Example 6: Identify as a special angles or not as a special angle, and decide in which quadrant the terminal sides of the angle is:

a) 1024^0 b) 940^0 c) 8100^0 d) $7\pi/5$

e) $35\pi/6$

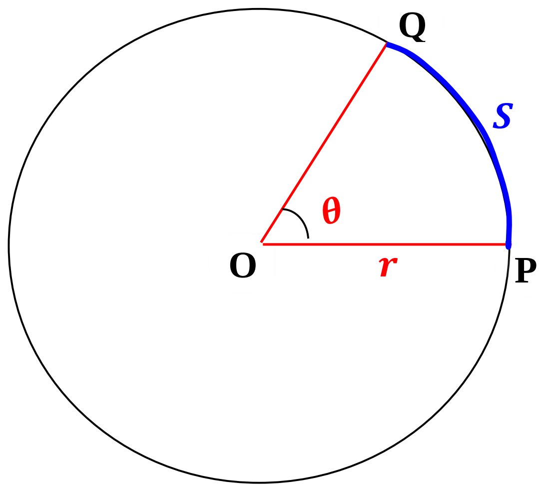
Example 7: Worked out Example (Reading HW): Page 702 Example 10.1.3

Examples YouTube Video: Angles in standard positions, quadrantal and co-terminal angles



Length of a Circular Arc and Area of a Circular Sector

Objective: We want to calculate the length of **arc** S and the **area** of **sector** OPQ of the circle of radius r shown below.



Length of a circular arc:

The circumference **C** of a circle of radius r is given by the formula $C = 2\pi r$

- In a circle of radius **r**, the length **S** of an arc that subtends a central angle of measure θ radians is $S = r\theta$

Area of a Circular Sector:

Area **A** of a circle of radius **r** is given by the formula $A = \pi r^2$

- In a circle of radius **r**, the area **A** of a sector with a central angle of measure θ radians is $A = \frac{1}{2}r^2\theta$

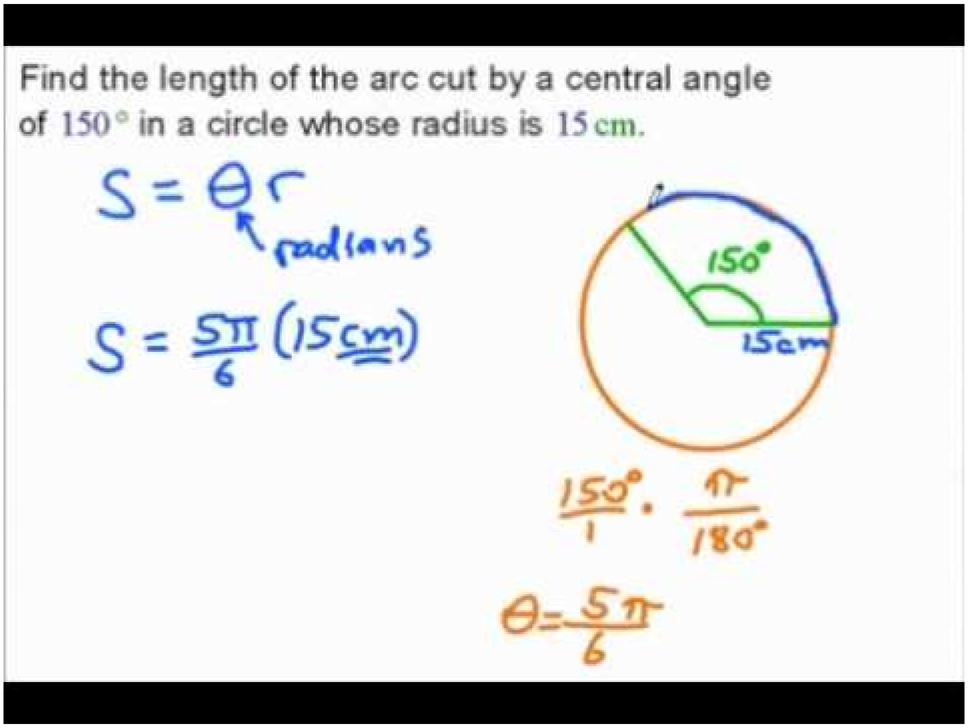
Example 1: In a circle of radius **2** find:

i. The length of the arc that subtends a central angle of measure a) $\frac{3}{4}\pi$ rad, b) 120^0

ii. The area of the sector that corresponds to a central angle of measure a) $\frac{3}{4}\pi$ rad, b) 120^0

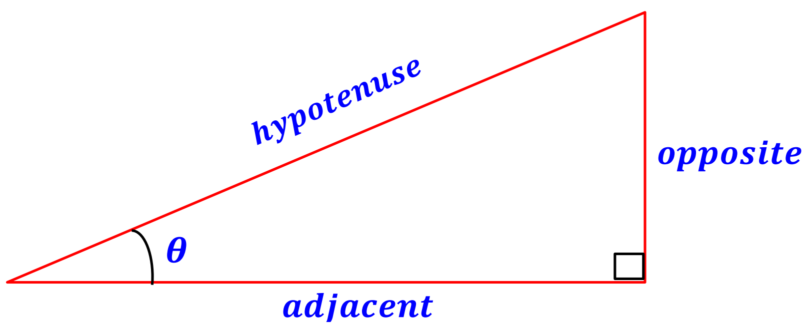
(Book 1) Homework:
Exercise 10.1.2 page 709: #1 – 41 and 56 – 63

Example YouTube video: Area of a sector and arc length



Trigonometric Ratios for Right Angle Triangles
Book 2 Chapter 1 page 1 – 23

For the angle **θ** in a **right-angled triangle** as shown, we name the sides as:



- **hypotenuse** (the side opposite the right angle)
- **adjacent** (the side "next to" **θ**)
- **opposite** (the side furthest from the angle **θ**)

We **define** the three **trigonometric ratios: sine θ**, **cosine θ**, and **tangent θ** as follows (we normally write these in the shortened forms $sin\ \theta$, $cos\ \theta$, and $tan\ \theta$)

$$sin\ \theta = \frac{opposite}{hypotenuse} \qquad cos\ \theta = \frac{adjacent}{hypotenuse} \qquad tan\ \theta = \frac{opposite}{adjacent}$$

The Reciprocal Trigonometric Ratios

Often it is useful to use the reciprocal ratios, depending on the problem. (Informally, the reciprocal of a fraction is found by turning the fraction upside down.)

Cosecant θ is the reciprocal of **sine θ**, **Secant θ** is the reciprocal of **cosine θ**, and **Cotangent θ** is the reciprocal of **tangent θ.** We usually write: $csc\ \theta$ for $cosecant\ \theta$ (In some textbooks, "csc" is written as "$cosec$"), $sec\ \theta$ **for** $secant\ \theta$ **and** $cot\ \theta$ **for** $cotangent\ \theta$, which are defined by:

$$\csc\theta = \frac{hypotenuse}{opposite} \qquad \sec\theta = \frac{hypotenuse}{adjacent} \qquad \cot\ \theta = \frac{adjacent}{Opposite}$$

Example 2:

a) The two legs of a right angled triangle are 3 and 4 units long; find the measure of all angles and the length of the hypotenuse.
b) One leg of a right triangle and its area are 5 units and 60 sq. units respectively. Find the other leg the hypotenuse and measures of all angles of the triangle.

Example 1: Worked out Example (Reading HW): Page 5 Example 1.3 and 1.4

Example 2: Worked out Example (Reading HW): Page 8 Example 1.5

Finding Angles, Given the Trigonometric Ratios

We are now going to work the **other way around**. We may know the final trigonometric ratio, but we don't know the original angle.

Example 1: Find θ, given that $\boldsymbol{tan\ \theta = 0.3462}$ and that $\mathbf{0° \le \theta < 90°}$.

Solution: We need to use the **inverse tangent function**. Our **answer** will be an **angle**.

So we use the **"$\tan^{-1}$" button** on our **calculator**, and we have:

$$\boldsymbol{\theta = tan^{-1}(0.3462) = 19.096°.}$$

Check: We can use our calculator to check our answer: $\boldsymbol{tan(19.096°) = 0.3462}$

Note: It is very common (and better) to use **"$arctan$"** instead of **"$\tan^{-1}$"**. It helps us to remember the difference. In the above example, we would write: $\boldsymbol{\theta = arctan(0.3462) = 19.096°}$.

We also write also **"$arcsin$"** for **"$\sin^{-1}$", "$arccos$"** for **"$\cos^{-1}$", "$arccsc$"** for **"$\csc^{-1}$"** etc.

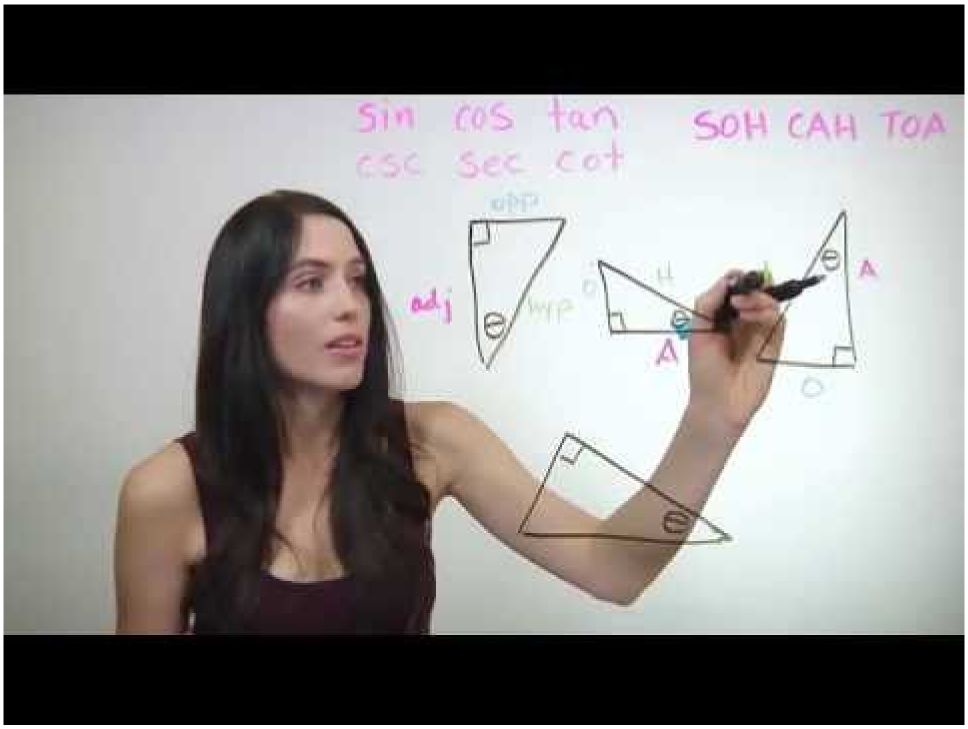
Exercise 2: **Find θ $(0° \leq \theta < 90°)$** given that

a. $sin\ \theta\ =\ 0.6235$ c. $tan\ \theta\ =\ 3.689$

b. $csc\ \theta\ =\ 8.32$ d. $sec\ \theta\ =\ 6.96$

Homework: (Book 2) Exercise 1.1 page 5: #1 – 13

Example YouTube: Trigonometry of the right triangles

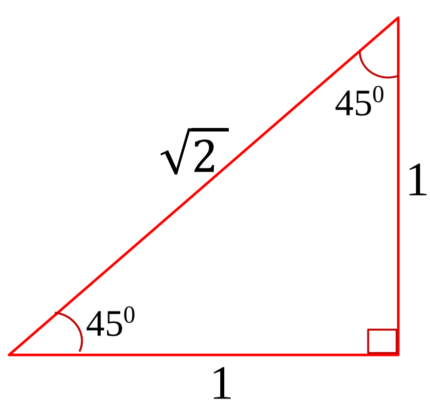


Special Triangles

1) **Isosceles Right Angled Triangle**
2) **A 30 – 60 – 90 triangle (Equilateral or Equiangular Triangles)**

1) Isosceles Right Angled Triangle or 45^0 – 45^0 triangle

Example 2: Worked out Example (Reading HW): Page 9 Example 1.6

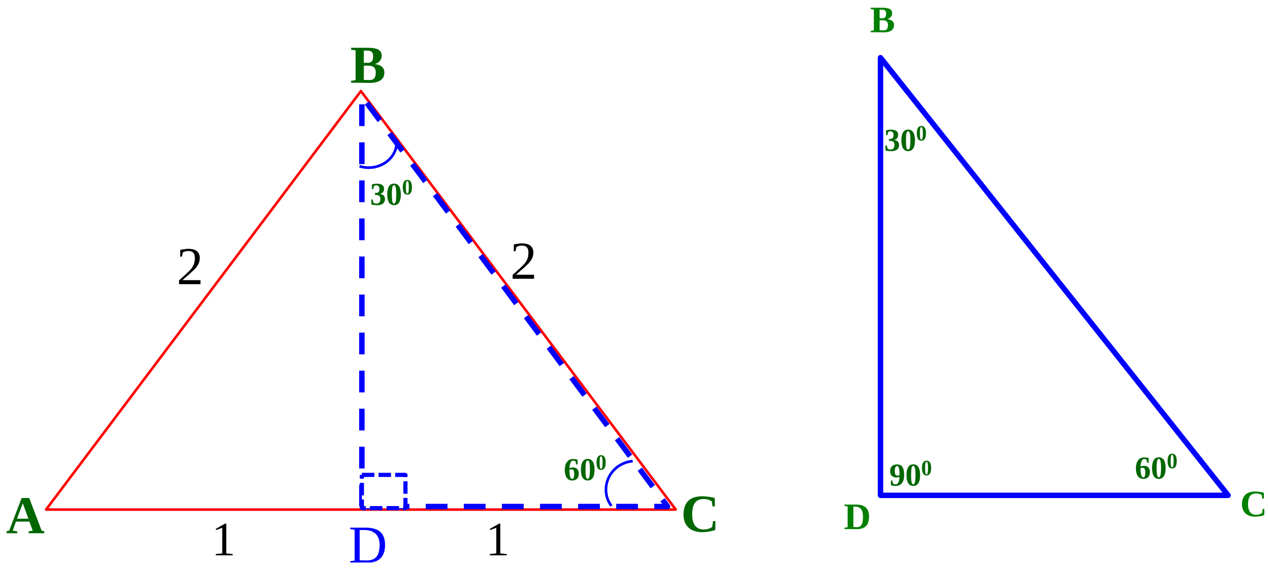


2) **A 30^0 - 60^0 - 90^0 triangle;** consider the equilateral triangle ΔABC of side 2 units. Point D is the foot of the foot of the perpendicular from vertex **B** to side **AC**. Triangle **BDC** is a **30 – 60 – 90** angle triangle with right angle at **D**.

BD is the height of the triangle and from Pythagoras Theorem $BD = \sqrt{3}$ units long

Measure $angle(BCD) = 60^0$ **and Measure** $angle(CBD) = 30^0$

Example 3: Worked out Example (Reading HW): Page 10 Example 1.7



Example 4: Worked out Example (Reading HW): Page 10 Example 1.8

Using 1) and 2) we construct the Table below

Table of values of trigonometric ratios for special angles

θ **in degree measure**	θ **in radian measure**	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30^0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45^0	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60^0	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

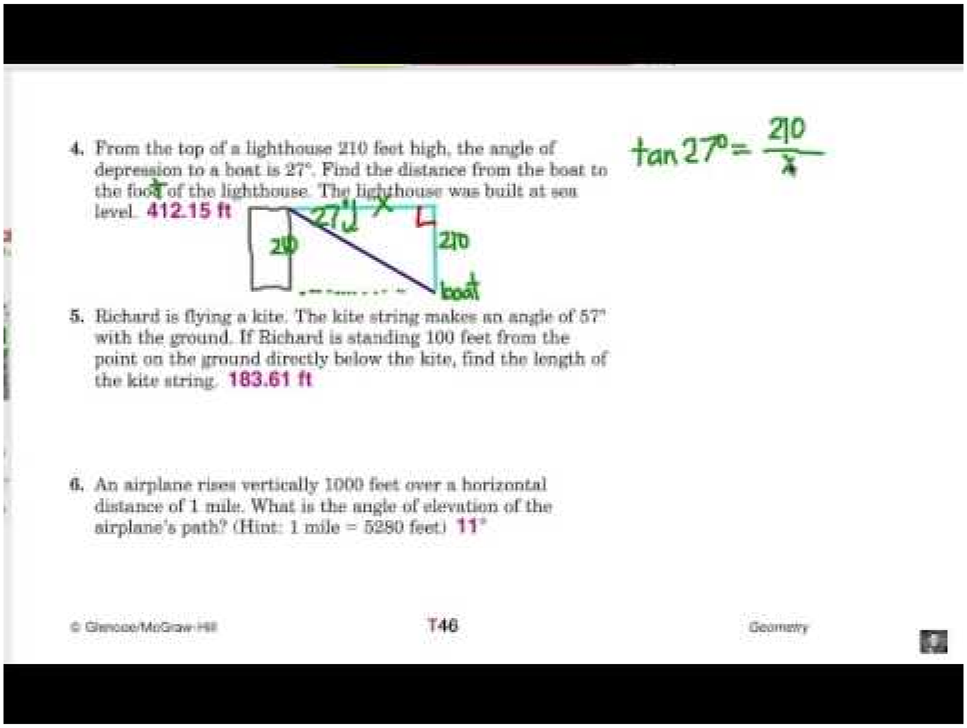
(Book 2) Homework Exercise 1.2 page 12: 1 – 34 odd numbers

The Right Triangle and its Applications (page 14 Book 2)

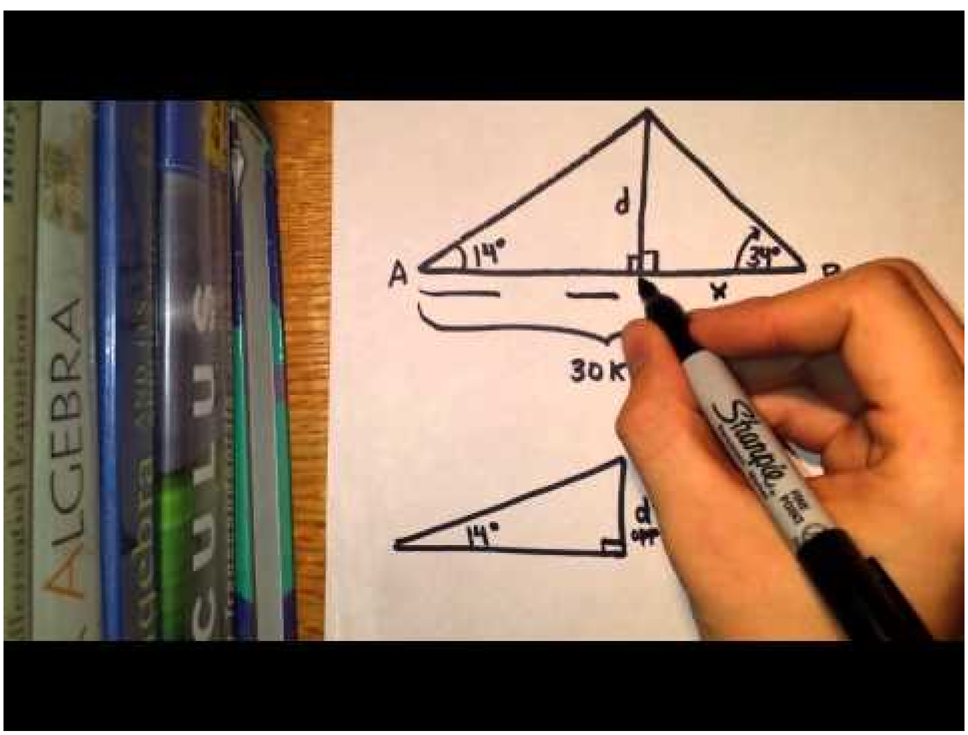
Many problems involve right triangles. We often need to use the trigonometric ratios to solve such problems.

Example: YouTube Videos

1)

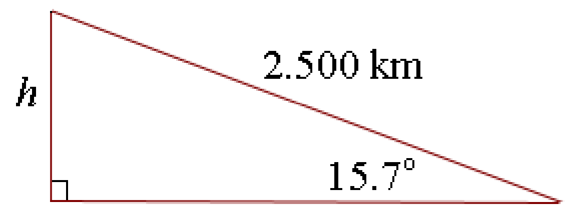


2)



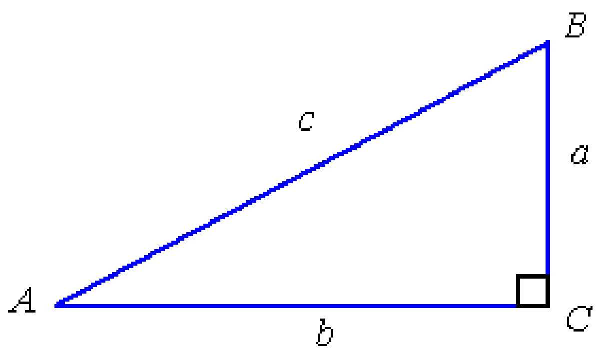
Example 1: Worked out Example (Reading HW): Page 14 - 19 Example 1.11 – 1.19

Example 1: Find the height h



Solving Right Triangles

A triangle has seven parts, **three sides, three angles and area**. Given almost any three of them; three sides, two sides and an angle, or one side and two angles; you can **find the other values**. This is called **solving the triangle.**



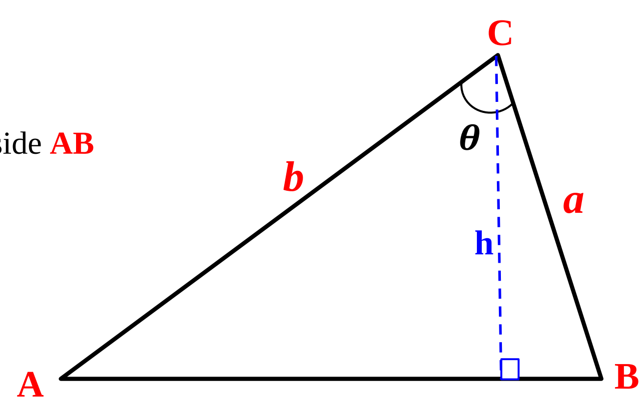
Example 2: Given an acute angle and one side; solve the right triangle ABC if angle A is 36° and side c is 10 cm.
Solution: Since angle A is 36°, then angle B is 90° − 36° = 54°.
To find an unknown side, say a, proceed as follows:

Example 3: Solve right triangle **ABC**, given that $\mathbf{A = 30^0}$ and $\boldsymbol{a} = \mathbf{3cm.}$

Example 4: Show that the **area A** of a **triangle** with side a, b and included angle θ **is given by**

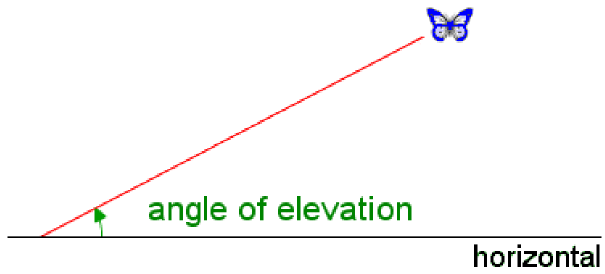
$$A = \frac{1}{2}ab\sin\theta$$

Solution: Draw height **h** from **C** to side **AB**

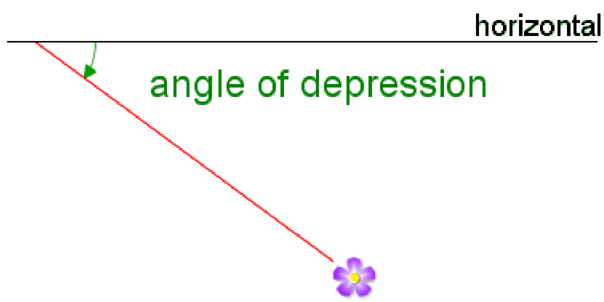


Angles of Elevation and Depression

In surveying, the **angle of elevation** is the angle **from** the **horizontal** looking **up** to some object:

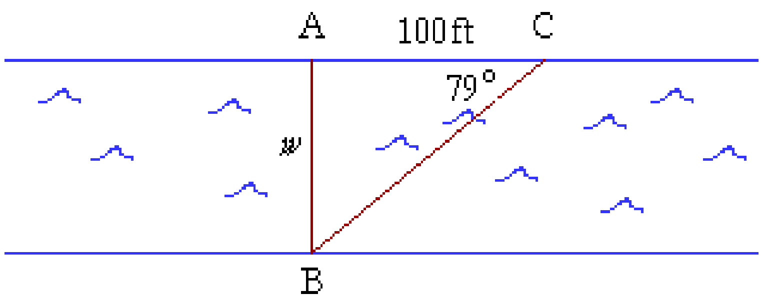


The **angle of depression** is the angle **from** the **horizontal** looking **down** to some object:



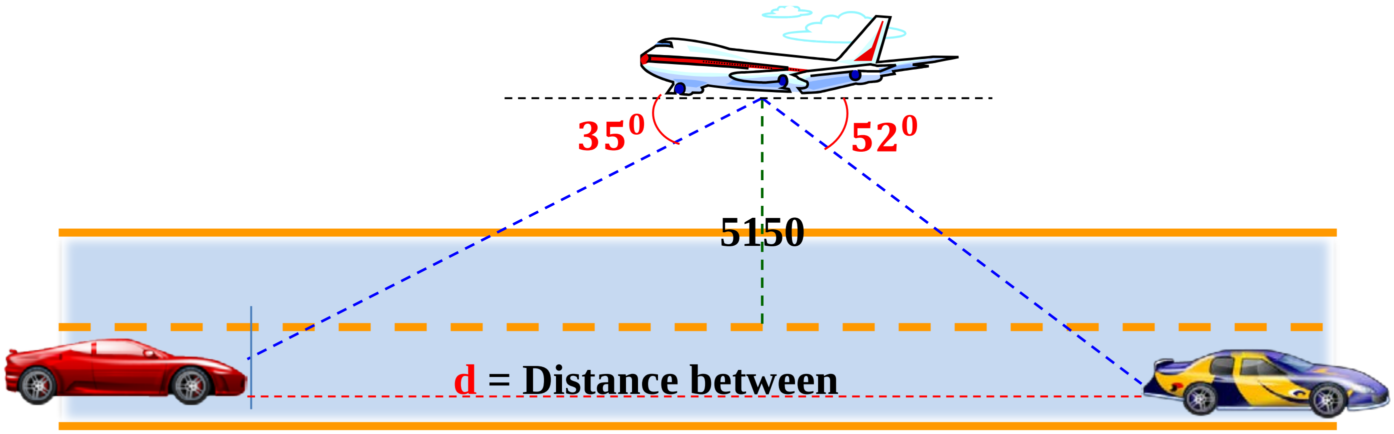
Example 5: The angle of elevation of an airplane is 23°. If the airplane's altitude is 2500 m, how far away is it?

Example 6: Two trees stand opposite one another, at points A and B, on opposite banks of a river.



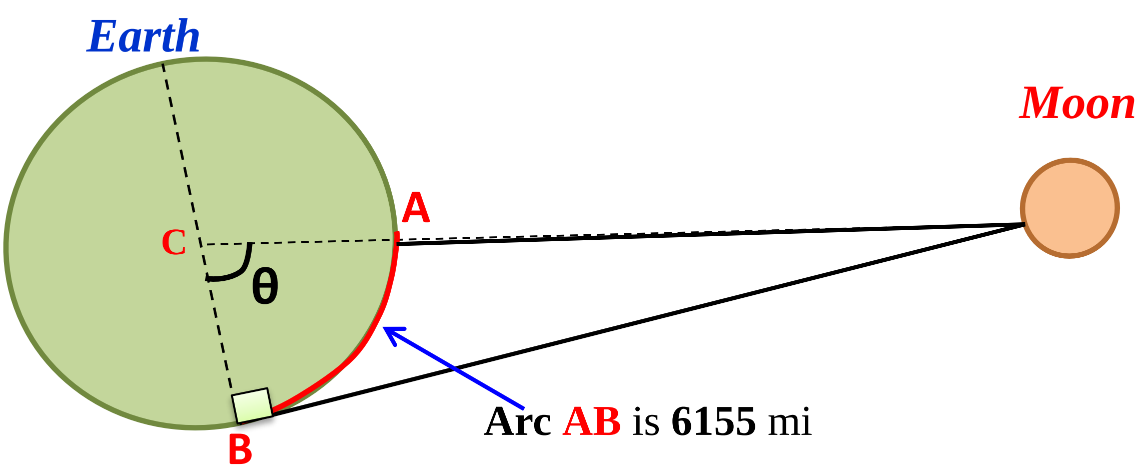
Distance **AC** along one bank is perpendicular to BA, and is measured to be 100 feet. Angle ACB is measured to be **79°**. How far apart are the trees; that is, what is the width *w* of the river?

Example 7: An **Airplane** is flying at an elevation of **5150** ft. directly above a straight highway. Two motorists are driving **cars** on the highway on opposite sides of the plane, the **angle** of **depression** to one car is 35^0 and to the other is 52^0, see figure. How far apart are the cars?



Example 8: Distance to the Moon: When the moon is seen at the zenith at a point **A** on Earth, it is observed to at the horizon from point **B**. Point **A** and **B** are **6155** mi apart, and the **radius** of the earth **CA = CB** is **3960** mi.

a) Find the angle θ in degree
b) Estimate the distance from point **A** to the moon



(Book 2) Homework Exercise 1.3 page 20: #1 – 29 odd numbers (Book 2)

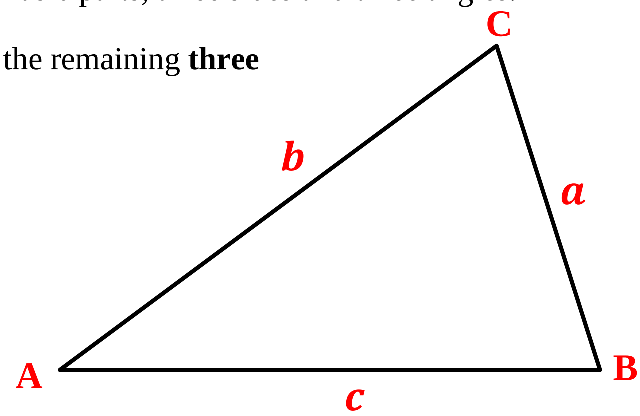
Chapter 11 (Book 1) (Page 806)
The Laws of Sines and Cosines

Objectives: By the end of these sections students should be able to:

- State the Sine and Cosine Laws
- Solve problems by using the Sine and Cosine Laws
- Solve triangles provided any three parts of a triangle
- Solve application problems

Consider Triangle **ABC** below; Triangle **ABC** has 6 parts, three sides and three angles.

Given any **three parts,** we want to **determine** the remaining **three**



The Law of Sines

The Law of Sines says that in any triangle the length of the sides are proportional to the sines of the corresponding opposite angles.

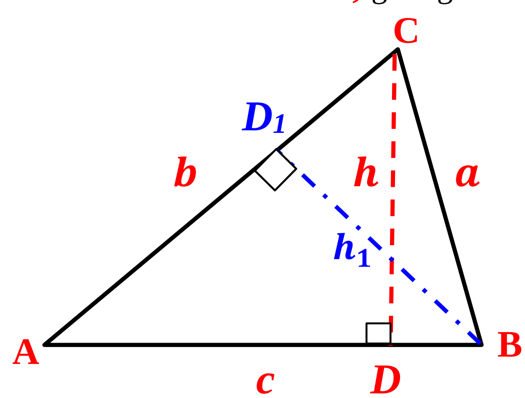
1) In any triangle **ABC** we have: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Proof: In $\triangle ABC$ below $h = CD$ is the **height** from vertex **C.** From Trig ratios $\sin A = \frac{h}{b}$, which gives $h = b \sin A$. Similarly $CD = h = a \sin B$; which implies $b \sin A = a \sin B$, giving $\frac{\sin A}{a} = \frac{\sin B}{b}$. Now let $BD_1 = h_1$ is the **height** from **vertex B**

And $h_1 = c \sin A = a \sin C$, which gives $\frac{\sin A}{a} = \frac{\sin C}{c}$

Combining results we obtain: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

This proves the **sine law**



Example: Worked out Example (Reading HW): Page 898 Example 11.2.2

Example: Worked out Example (Reading HW): Page 903 Example 11.2.4

(Book 1) Homework Page 904 Exercises 11.2.1: 1 – 20 odd numbers

The Law of Cosines (Page 910)

The Law of Sines cannot be used directly to solve triangles if we know all three sides or two sides and the angle included between them is given. In these cases we use the Law of Cosines.

2) Given any **triangle ABC**, see figure, the following holds:

a) $a^2 = b^2 + c^2 - 2bc\cos A$

b) $b^2 = a^2 + c^2 - 2ac\cos B$

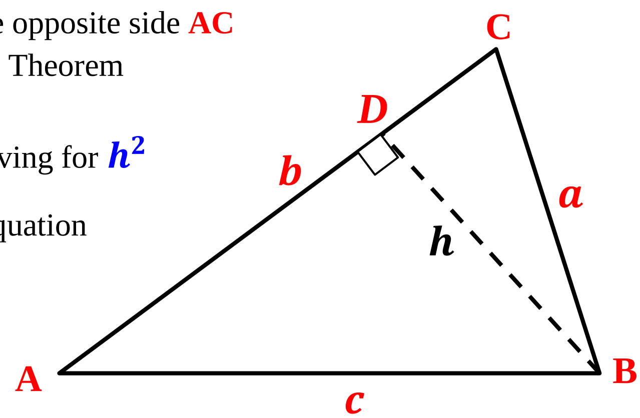
c) $c^2 = a^2 + b^2 - 2ab\cos C$

Proof: Two Cases

Note: The role Pythagoras law plays in the proof

i) **Acute angled** triangle, see figure

In **△ABC** drop perpendicular **BD** from angle **B** to the opposite side **AC**
△ABD and △BCD are right triangles. By Pythagoras Theorem



$c^2 = h^2 + (AD)^2$ and $a^2 = h^2 + (DC)^2$, solving for h^2 in the 2nd equation and replacing h^2 the in the 1st equation

we get, $c^2 = [a^2 - (DC)^2] + b^2$, but

$DC = a\cos C$, and $AD = b - DC$

which in turn gives

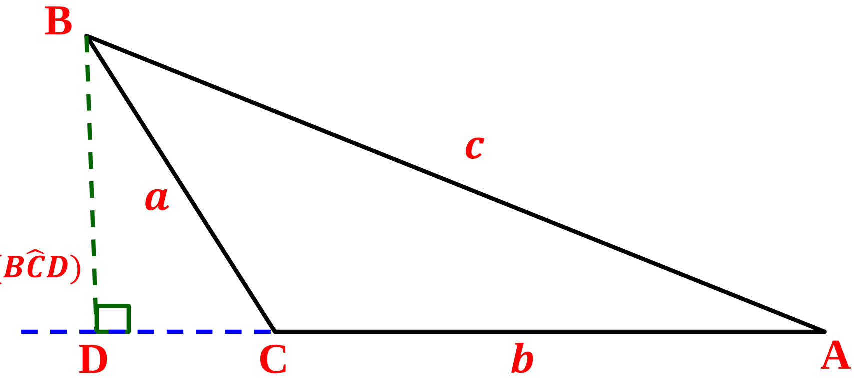
$$c^2 = [a^2 - (a\cos C)^2] + (b - a\cos C)^2$$

$$c^2 = a^2 - (a\cos C)^2 + b^2 - 2ab\cos C + (a\cos C)^2$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \#$$

We prove **a) and b**) in a similar way

ii) **Obtuse angled** triangle, see figure **Angle** **C** is the obtuse angle. Drop perpendicular from vertex **B** to side **AC** to get **BD**



Now $C^2 = (BD)^2 + (b + DC)^2$

But $BD = a\sin(B\widehat{C}D)$ and $DC = a\cos(B\widehat{C}D)$

Replacing **BD** and **DC** gives

$$C^2 = (a\sin(BCD))^2 + (b + a\cos(BCD))^2$$

$$C^2 = a^2 sin^2(BCD) + b^2 + 2ab\ cos\ (BCD) + a^2 cos^2(BCD)$$ [**BCD** is the reference angle of **BCA**]

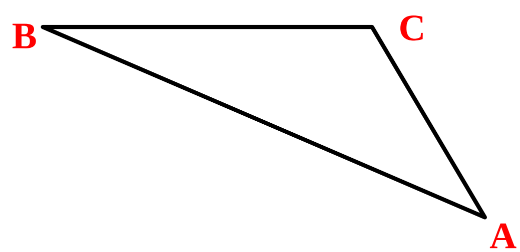
$$C^2 = a^2\left(sin^2(BCD) + cos^2(BCD)\right) + b^2 - 2ab\ cos(BCA)$$

$$C^2 = a^2 + b^2 - 2ab\ cos(BCA) \qquad \#$$

Example: Worked out Example (Reading HW): Page 911 Example 11.3.1

Example: Worked out Example (Reading HW): Page 913 Example 11.3.2

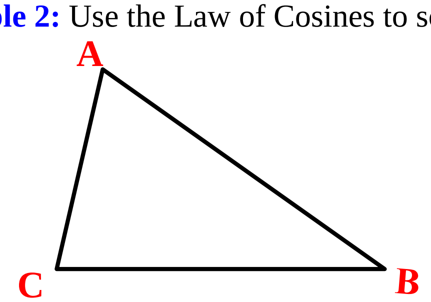
Example 1: Use the Law of Sines to solve the triangles



a) **Let BC = 17, $\widehat{A} = 37.5^0$ and $\widehat{B} = 28.1^0$**

b) **Let $\widehat{C} = 120^0$, AB = 45, and BC = 36**

Example 2: Use the Law of Cosines to solve the triangles



a) $\mathbf{a = 3, b = 4}$**, and** $\widehat{C} = 53^0$

b) $\mathbf{b = 60, c = 4}$**, and** $\widehat{A} = 70^0$

c) **a = 73.5, $\widehat{B} = 61^0$, and $\hat{c} = 83^0$**

Example 3: The leaning Tower of Pisa: The bell tower of the cathedral in Pisa, Italy, leans 5.6^0 from the vertical. A tourist stands 105 m from its base, with the tower leaning directly toward her. She measures the angle of elevation to the top of the tower to be 29.2^0. Find the height of the tower to the nearest meter.

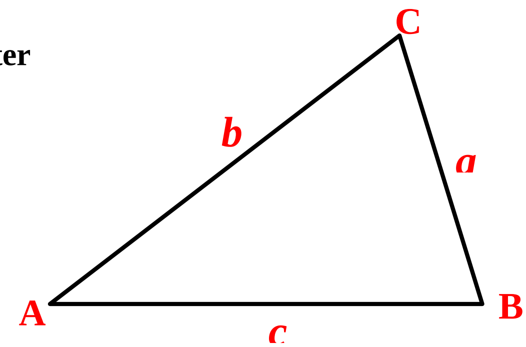
Example 4: A triangular field has sides of lengths 22, 36, and 44 yard. Find the largest angle

Example 4: (The Area of a Triangle, **Heron's Formula Page 914 – 915)**

The area A of triangle **ABC** with sides of length ***a, b,*** and ***c*** is given by:

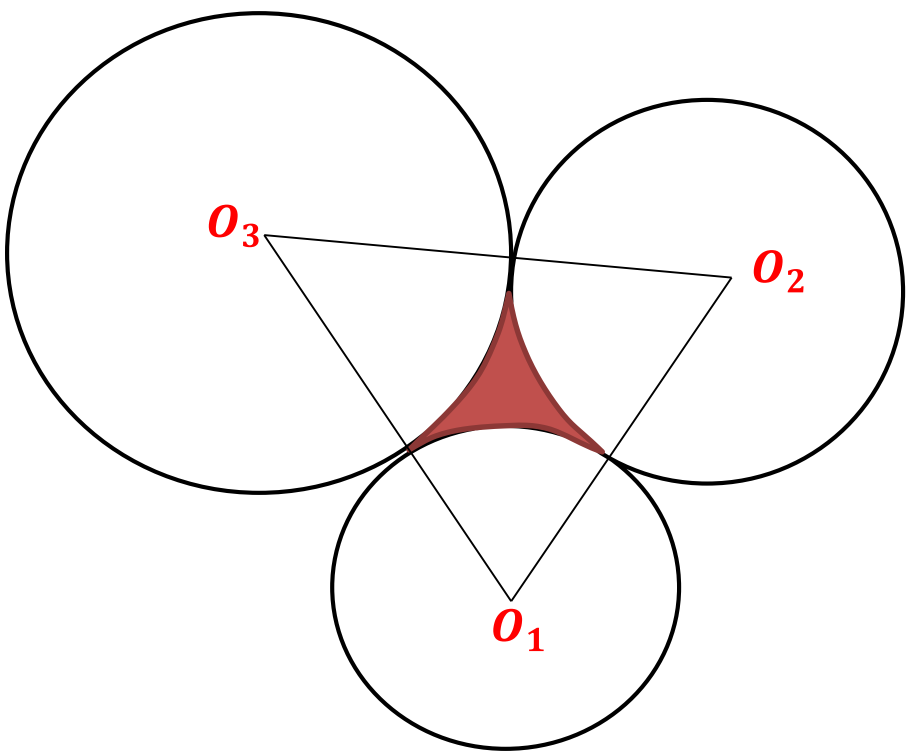
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = \frac{1}{2}(a+b+c)$; that is, s is **half** the **perimeter**



Example 4: Worked out Example (Reading HW): Page 915 Example 11.3.3

Example 5: The **three circles** of radius 4, 5, and 6 units are mutually tangent to one another. Find area of the **shaded region** enclosed between the circles, see figure below; O_1, O_2, *and* O_3 are centers.

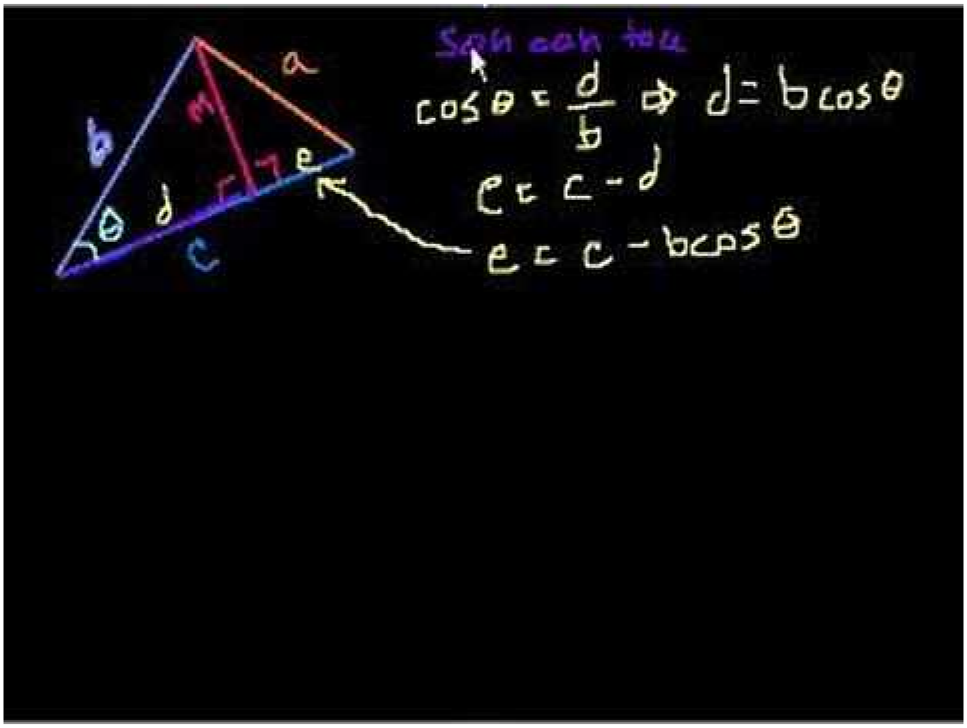


(Book 1) Homework

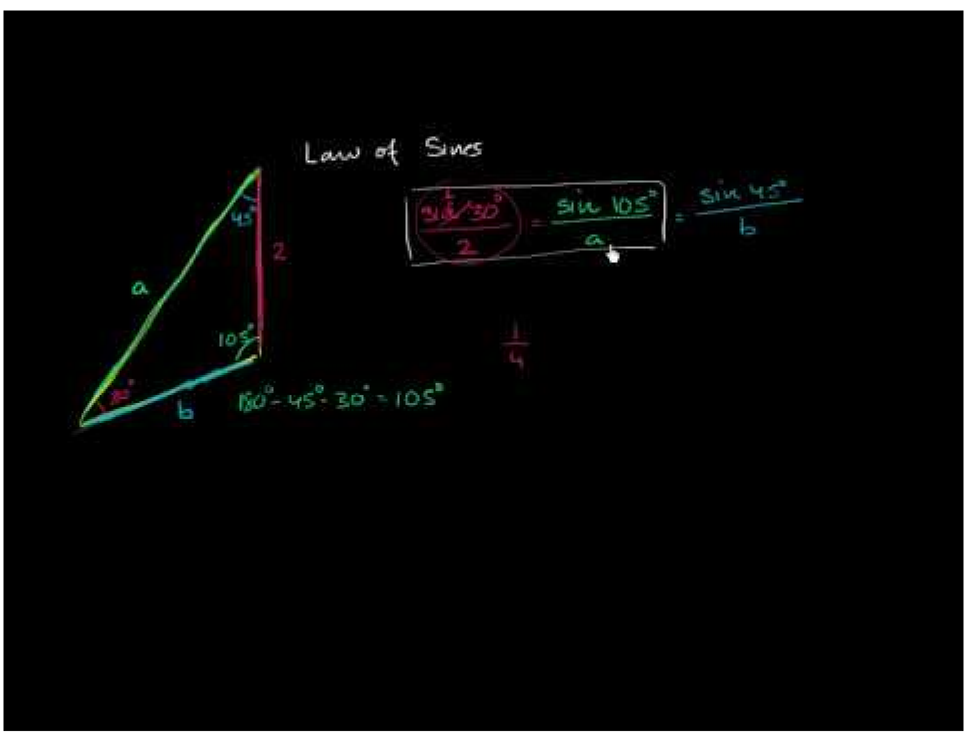
Exercises 11.3.1 Page 916: 1 – 21 Odd Numbers

Examples: Video links from YouTube:

- **Sine and cosine rules:** https://www.youtube.com/watch?v=I8LI7wPSvNI
- **The law of cosines:** https://www.youtube.com/watch?v=ZElOxG7_m3c



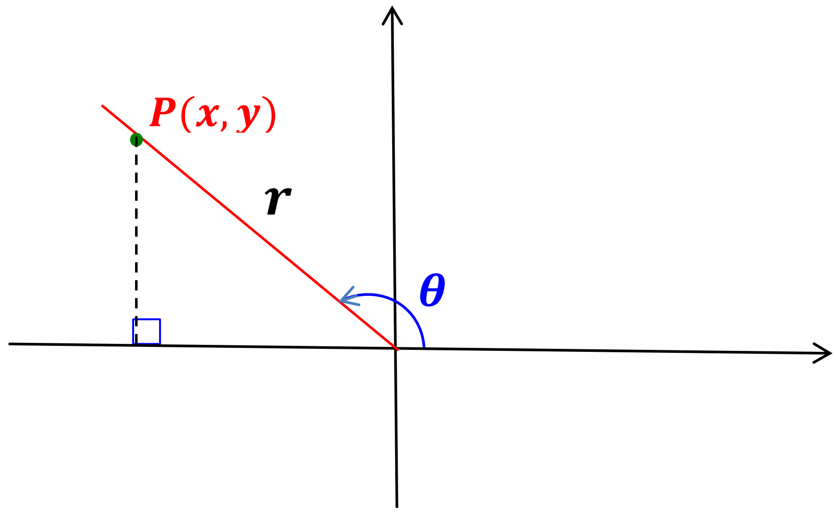
- **The Law of sines:** https://www.youtube.com/watch?v=VjmFKle7xIw



Trigonometric Functions of Angles

Definition (Trigonometric Functions)

Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ (See figure below).



If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then the **six the trigonometric functions** of θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}, x \neq 0$$

$$\csc\theta = \frac{r}{y}, y \neq 0 \qquad \sec\theta = \frac{r}{x}, x \neq 0 \qquad \cot\theta = \frac{x}{y}, y \neq 0$$

Note:

a) θ could **have any** real number for a **measure**

b) The **values** of the **trigonometric functions** do **not depend** on the choice of the point $P(x, y)$ on the **terminal side** of θ

Example: YouTube video link:

- **Trig Function of any angle**: https://www.youtube.com/watch?v=7imLjtXic7k
- **Trig Function of any angle:** https://www.youtube.com/watch?v=5cRnf6Ov38U

Examples 1: Finding Exact Values of Trigonometric Ratios

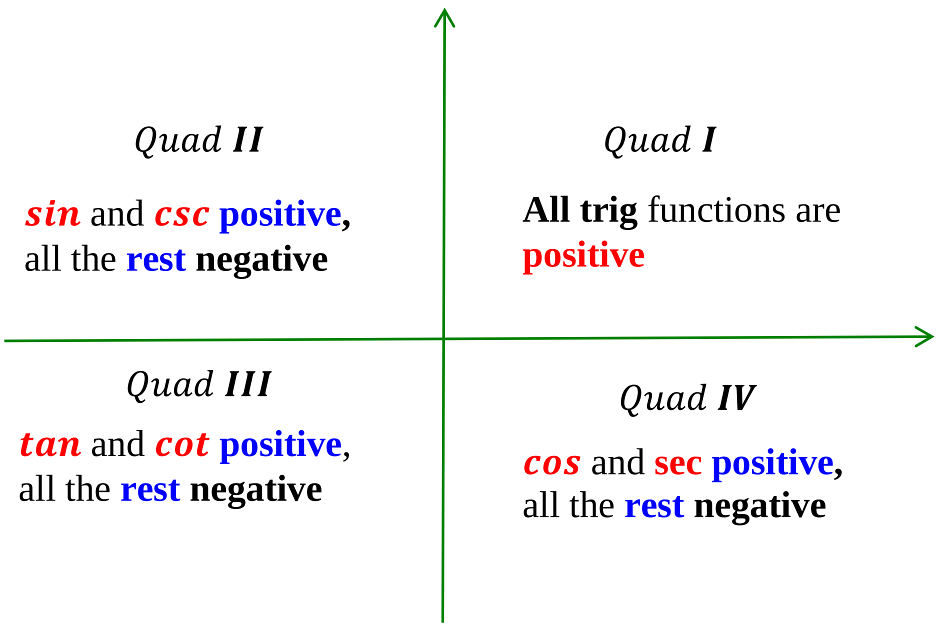
Find the **exact values** indicated; this means don't use your calculator to find the values (which will normally be a decimal approximation). Keep everything in terms of surds (square roots).

1. Let $P(x, y)$ be a point on the terminal side of angle θ. Find the **exact trigonometric values** of θ for:

a) $P(-2,\ 3)$ **b)** $P(-3, -1)$ c) $P(3,\ 4)$ d) $P(0,\ 3)$ e) $P(-5,\ 0)$

2. Find the exact value of $\boldsymbol{\sin\theta}$ if the terminal side of **θ** passes through **(7, 4).**

Sign of Trigonometric Functions

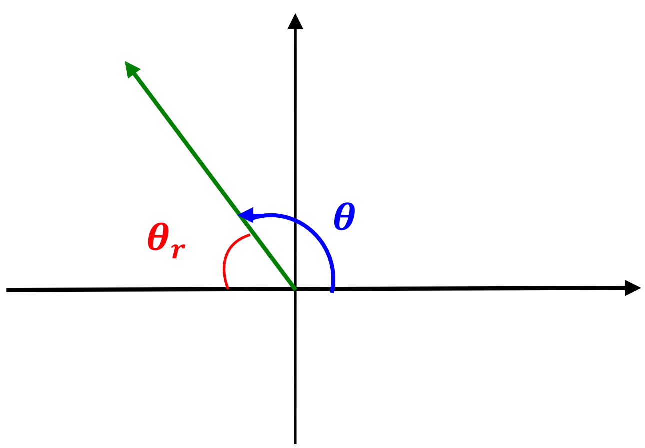


Example 2: For each of the following points find the **sign of the trigonometric functions**. Provided each point is on the terminal side of some angle

a) $P(4,\ 5)$ b) $Q(-3,\ -6)$ c) $P(45,\ -100)$ d) $Q(-13,\ 12)$

Reference Angles (page 717) (Book 1)

Definition: In the standard position, the **reference angle** $\boldsymbol{\theta_r}$ is the **acute angle** formed between the **terminal** side of $\boldsymbol{\theta}$ and the ***x*-axis**. See Figure Below



Example 4: Draw each of the following angles in the standard position and find the corresponding reference angle for each.

a) -510^0 c) 310^0 e) -210^0

b) $-13\pi/3$ d) $-5\pi/6$ f) $7\pi/4$

Trigonometric Functional Values of Any Angle

Using reference angles we can calculate the trigonometric functional values of any angle.
Important: Recall trig functions of special right triangles

Theorem (Reference Angle Theorem)

Suppose θ_r is the reference angle for θ. Then $\cos\theta = \pm\cos\theta_r$ and $\sin\theta = \pm\sin\theta_r$, where the choice of the $\pm$ depends on the quadrant in which the terminal side of θ is.

Note in general:

$$|\, Trig\ Values\ of\ angle\ \theta\ | = |\, Trig\ Values\ of\ angle\ \theta_r\ |$$

That means, the **values** of the **trig function** of angle θ **are the same** as **the trig values** of the **reference angles** θ_r of θ, give or take **a minus sign**.

Example 5: Homework Reading page 720 Example 10.2.2

Example 6: Find the cosine and sine of the following angles

a) 225^0 b) $\frac{11\pi}{6}$ c) $-7\pi/4$ d) $7\pi/6$

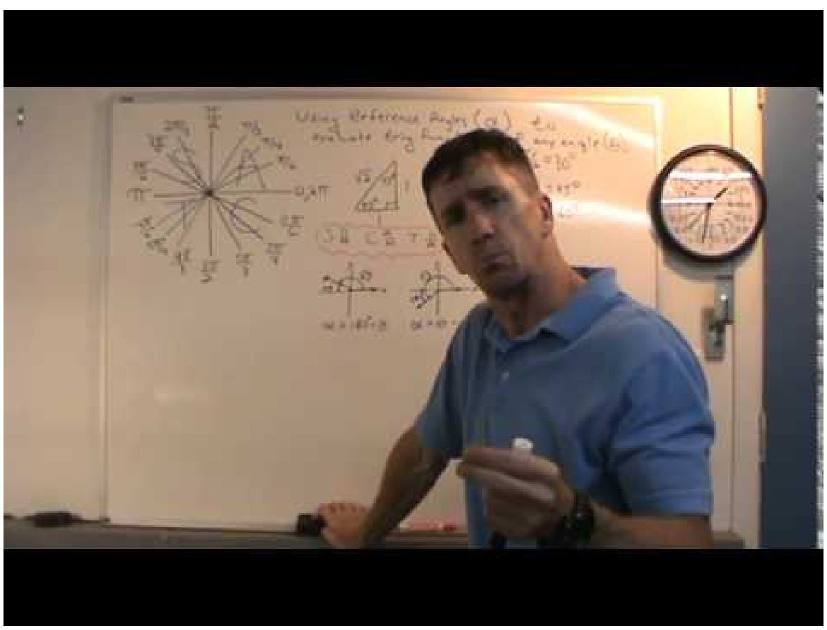
Example 6: Find all trigonometric functional values of:

a) $\sin 135^0$ b) $\tan 390^0$ c) $\sin 240^0$

d) $\cot 495^0$ e) $\sin\frac{12\pi}{3}$ f) $\sec(-\pi/4)$

Example: YouTube Video

- **Trigonometric functions of any angle:** https://www.youtube.com/watch?v=zSMVmow79Ko

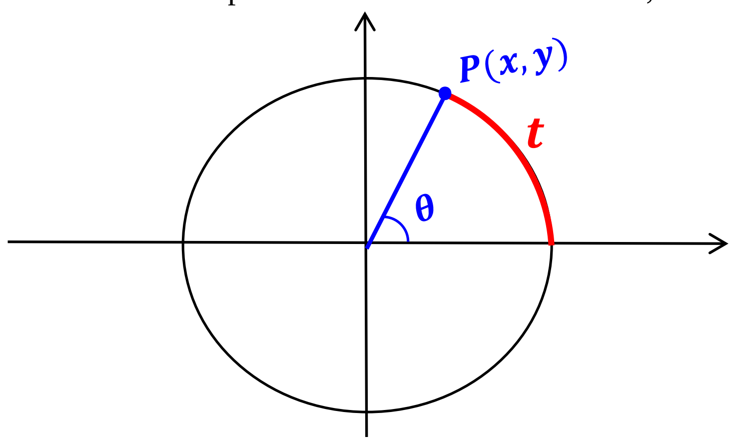


10.2 The Unit Circle (page 717) (Book 1)

The **unit circle** is the circle of **radius one** in the coordinate plane with center **(0, 0).**

The trigonometric functions are most easily understood in the context of a circle in the Cartesian plane, in which **angles** are always **measured** from the **positive *x* axis**: **positive angles** are measured in an **anti-clockwise direction**, and **negative angles** are measured in a **clockwise direction**.

Let t be any real number that corresponds to an arc on the unit circle, see figure below.



If t subtends a central angle of measure θ, then the trigonometric functions are defined as follows:

$\sin\theta = y$ $\qquad\cos\theta = x$ $\qquad\tan\theta = \frac{y}{x}, x \neq 0$

$\csc\theta = \frac{1}{y}, y \neq 0$ $\qquad\sec\theta = \frac{1}{x}, x \neq 0$ $\qquad\cot\theta = \frac{x}{y}, y \neq 0$

If θ is measured in radian, then $\theta = t$ giving:

$\sin t = y$ $\qquad\cos t = x$ $\qquad\tan t = \frac{y}{x}, x \neq 0$

Example 1: Show that the following points are on the unit circle and find the values of the six trig functions corresponding to the point **P** on the terminal side of an angle t:

a) $P\left(-\frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{2}\right)$ $\qquad$ b) $P\left(\frac{3}{5}, \frac{4}{5}\right)$

Example: Homework Reading, **page 717** Example 10.2.1

Example 2: Suppose the terminal side of angle in the standard position contains the point P(4/5, -3/5). Find the trigonometric functional values of the angle.

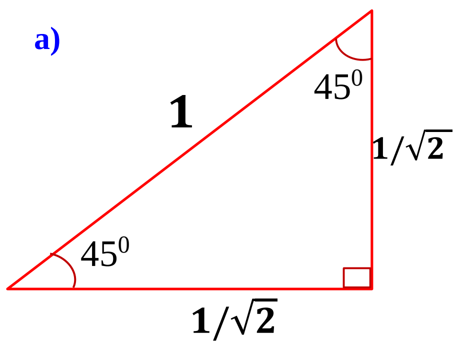
Example 3: Calculate the trigonometric functional values of the following angles

a) $\pi/2$ **b)** $-\pi/2$ c) $-\pi$ **d)** π

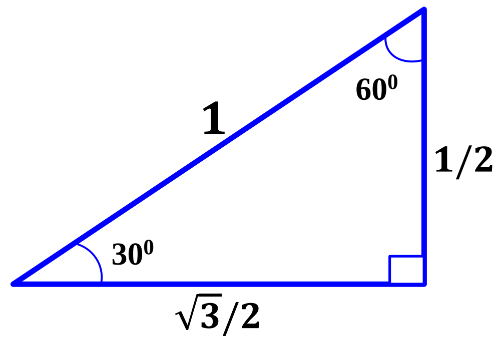
Example 4:

A) Use the **special right triangles** shown below to construct trig values of angles on a unit circle:

a)



b)



Special Angles on one complete rotation

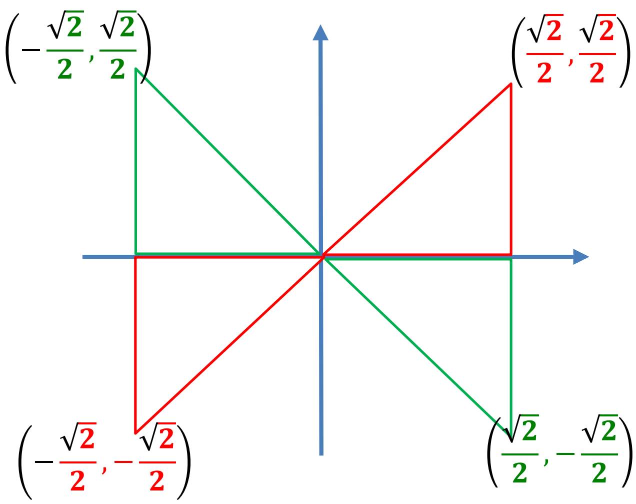
a) $\pm 30°$, $60°$, $45°$, $120°$, $135°$, $150°$, $210°$, $225°$, $240°$, $300°$, $315°$, and $\pm 330°$. $\pm 360°$.

b) $\pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{6}, \pm\frac{7\pi}{6}, \pm\frac{5\pi}{4}, \pm\frac{4\pi}{3}, \pm\frac{11\pi}{6}, \pm\frac{7\pi}{4}, \pm\frac{11\pi}{6}, \pm 2\pi$

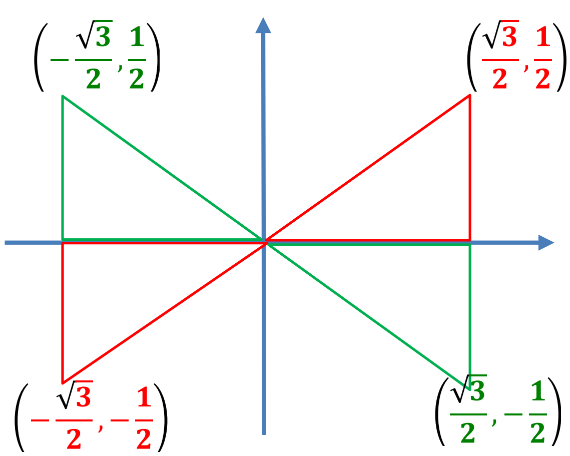
B) Construct table for these special angles

Solutions:

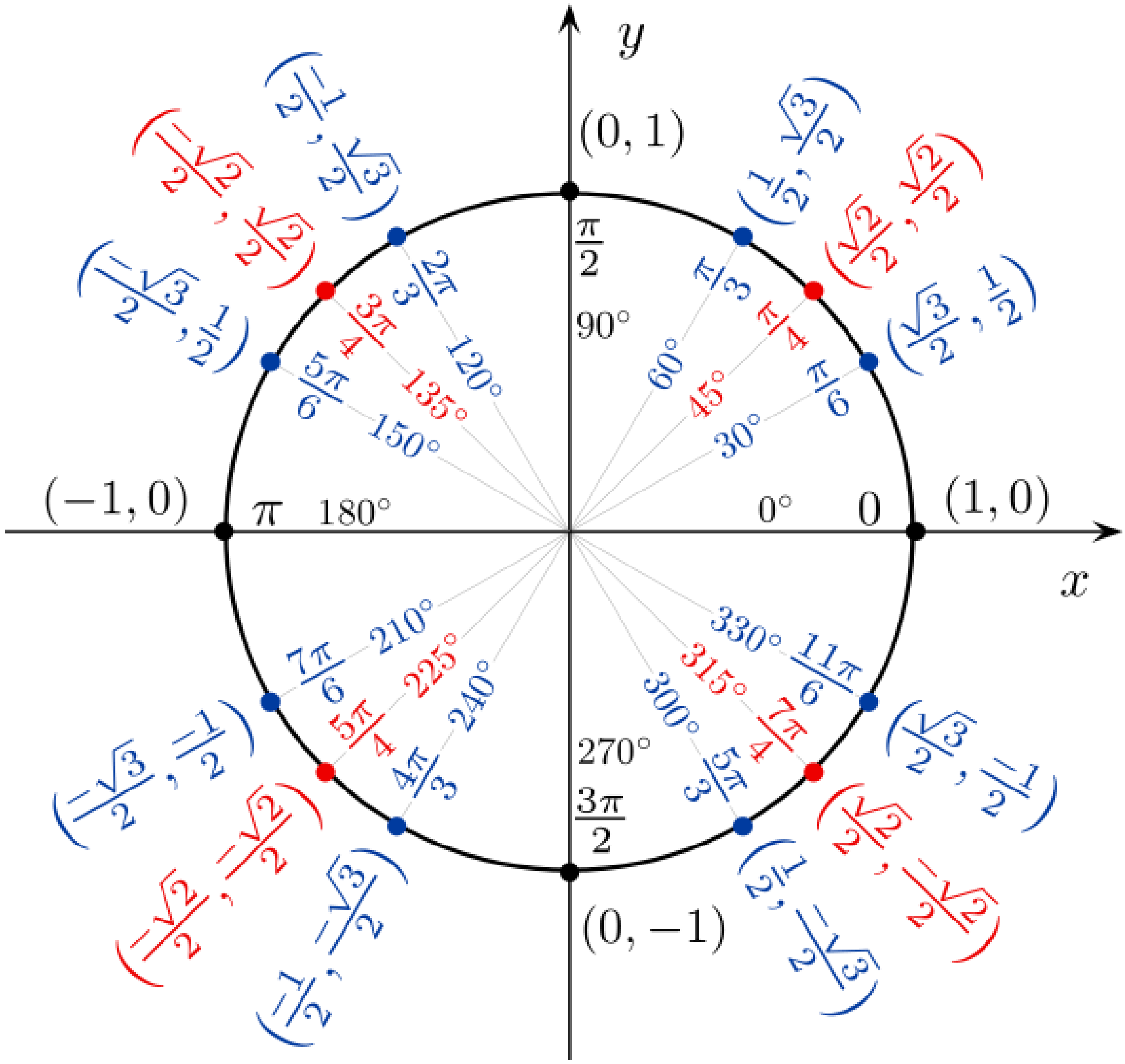
45^0 *by* 45^0 *and* 90^0 *triangles*



30^0 *by* 60^0 *and* 90^0 *triangles*



Solution: Pictorially



Example 5: Page 728 Example 10.2.5. Find all the angles which satisfy the given condition.

a) $\cos\theta = \frac{1}{2}$

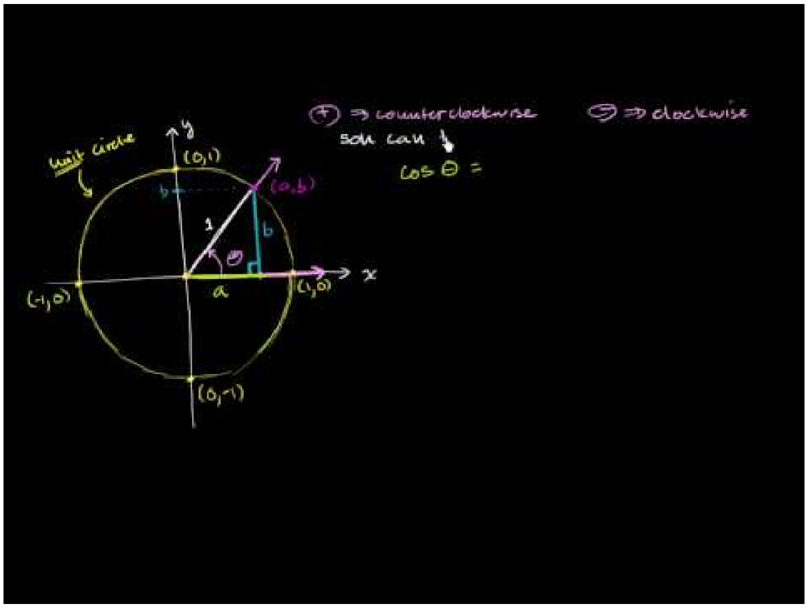
b) $sin\,\theta = -\frac{1}{2}$

c) $\cos\theta = 0$

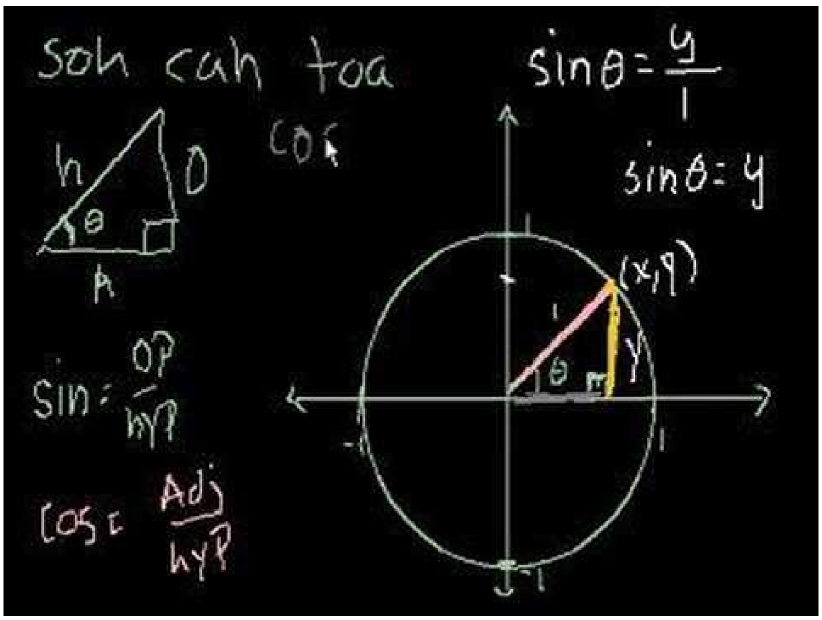
(Book 1) Homework Exercises 10.2.2 page 736: # 1 – 48 odd numbers, 55, 56, 57, and 58

Example: YouTube Video: The Unit Circle:

- **1) Introduction to the unit circle**



- **2) Unit circle definition of Trig functions**



Example: YouTube Video: Using the unit circle to evaluate trig values of angles

- https://www.youtube.com/watch?v=wGFOlLJz24I
- https://www.youtube.com/watch?v=dX972BmIiGU
- https://www.youtube.com/watch?v=GpmGBfR6Nus

Trigonometric Identities

(Book 2 page 65)

Objectives: By the end of this section student should be able to

- Identify Fundamental or Basic Identities
- Find trigonometric values using the trig Identities
- Evaluate trigonometric functions

Identities

Equations: Three types

1) **Conditional equations**: These types of equations have finitely number of solutions.
   Example: a) $2x - 5 = 7x$, b) $3x^2 - 4x - 6 = 0$
2) **Contradictions**: These are equations that do not have solutions
   Examples: $2x - 1 = 2(x - 1) + 6$
3) **Identities**: These types of equations hold true for any value of the variable
   Examples: $(x + 5)(x - 5) = x^2 - 25$

Trigonometric Identities are identities of the Trigonometric equations. We use an identity to give an expression a more convenient form. In calculus and all its applications, the trigonometric identities are of central importance.

Fundamental or Basic Trigonometric Identities

Reciprocal Identities, Quotient Identities and Pythagorean Identities

1) Reciprocal identities

$$\sin\theta = \frac{1}{\csc\theta}, \text{ and } \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta}, \text{ and } \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}, \text{ and } \cot\theta = \frac{1}{\tan\theta}$$

Proof: Follows directly from the definition of trig functions.

2) Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Proof: Follows directly from the definition of trig functions.

3) Pythagorean Identities

a) $\sin^2\theta + \cos^2\theta = 1$

b) $1 + \tan^2\theta = \sec^2\theta$

c) $1 + \cot^2\theta = \csc^2\theta$

Proof: a) Let $P(x, y)$ be on the terminal side of the angle θ.

Then $r = \sqrt{x^2 + y^2}$ which implies that $r^2 = x^2 + y^2$, $\sin\theta = \frac{y}{r}$, and $\cos\theta = \frac{x}{r}$

And so, $\sin^2\theta + \cos^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$

Note:

- From **a)** it follows that: $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$,
- **b)** and **c)** are similarly proved.
- $\sin^2\theta$, "**sine squared theta**", means $(\sin\theta)^2$

Example 1:

a) Express $\sin\theta$ in terms of $\cos\theta$
b) Express $\cos\theta$ in terms of $\sin\theta$
c) Express $\tan\theta$ in terms of $\cos\theta$, where θ in Quadrant II
d) If $\tan\theta = \frac{3}{2}$ and θ is in Quadrant III, find $\sin\theta$ and $\cos\theta$
e) If $\cos\theta = \frac{1}{2}$ and θ is in Quadrant IV, find all other trig values of θ
f) Use the basic trigonometric identities to determine the other five values of the trigonometric functions given that $\sin\alpha = 7/8$ and $\cos\alpha > 0$.
g) x is in quadrant **II** and $\sin x = 1/5$. Find $\cos x$ and $\tan x$.

Example 2: Prove the **Pythagorean Identities b)** and **c)**

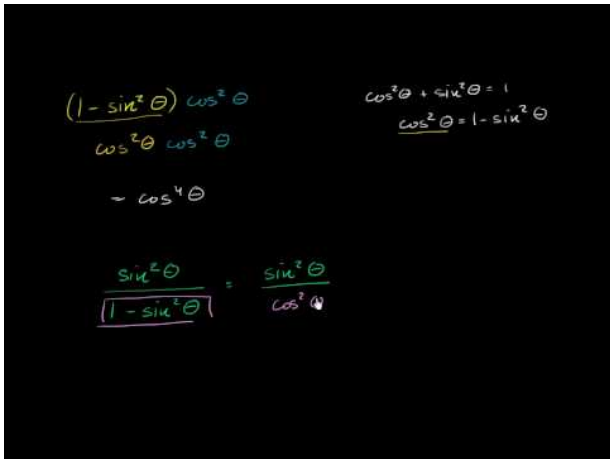
Example 3: Homework Reading page 67 – 69 Examples 3.1, 3.2, **3.3**, 3.4, **3.5**, 3.6, **3.7**

(Book 2) Homework Exercises 3.1 page 70: # 1 – 21 odd numbers

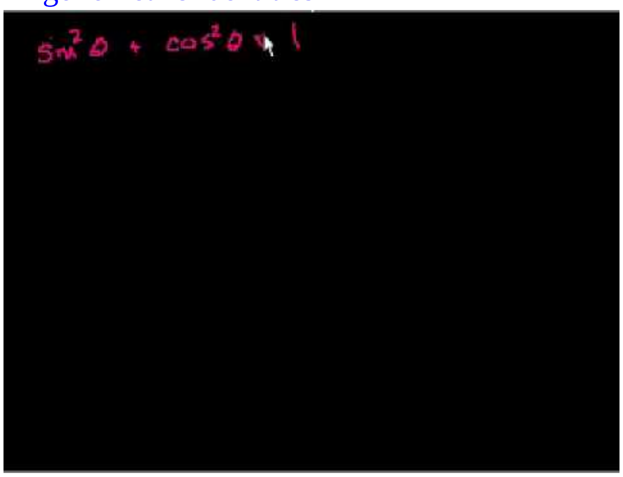
Examples YouTube Videos Trigonometric Identities

1) https://www.youtube.com/watch?v=iKWGv0xcuCA
2) https://www.youtube.com/watch?v=QGk8sYck_ZI
3) https://www.youtube.com/watch?v=raVGSdfBVBg

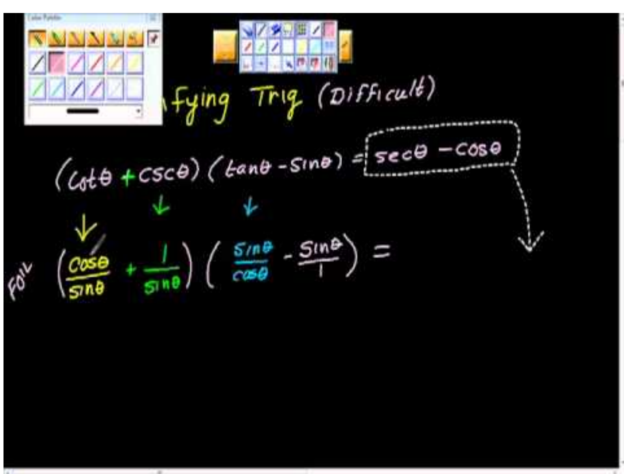
4) Pythagorean Identity



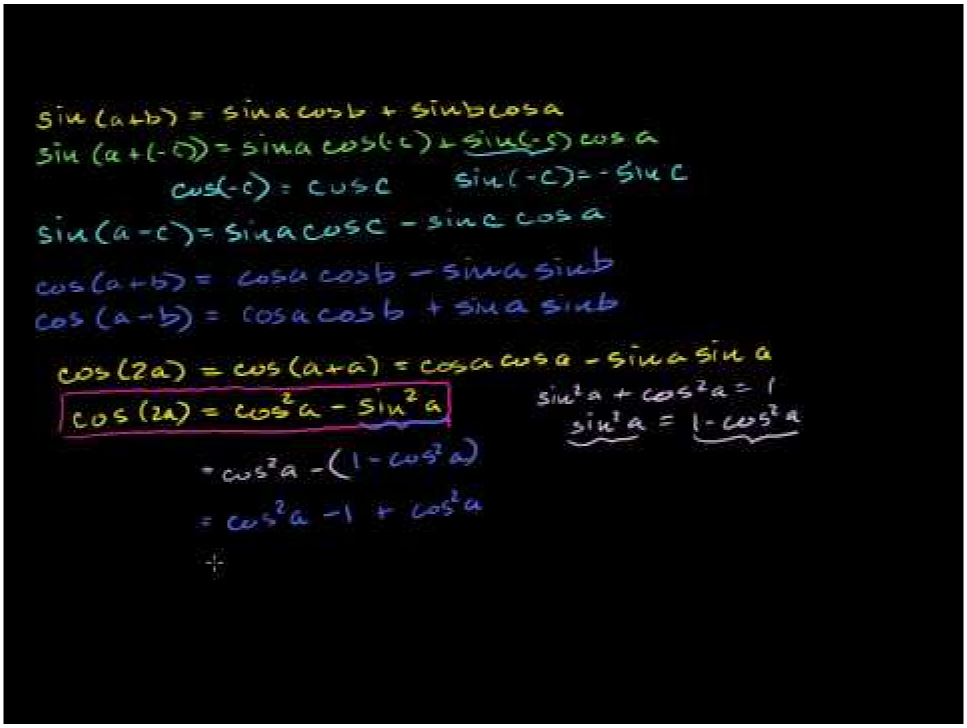
5) Trigonometric Identities



6) Verifying more difficult Trig. Identities



7) Review Trig identities (1)



8) Understanding Trig Identities